

Suppose G is a cyclic gp of order n

G is cyclic: $\exists x \in G$ so that

$$\bullet G = \{ \dots, x^{-2}, x^{-1}, 1, x, x^2, \dots \}$$

$$= \{ x^k : k \in \mathbb{Z} \}$$

$$\bullet |G| = n.$$

Ex $C_n = \{ \text{rotations of } \mathbb{R}^2 \text{ by an angle that is a multiple of } \frac{2\pi}{n} \}$.

the different rotations are rotations by $\frac{2\pi}{n} \cdot k \quad k \in \mathbb{Z}$

Propⁿ Suppose G is cyclic of order n , w/ generator x
 $\text{we } G = \{ x^k : k \in \mathbb{Z} \}.$

Then $G = \{ 1, x, x^2, \dots, x^{n-1} \}.$

Pf: Define $S = \{ k \in \mathbb{Z} : x^k = 1 \}.$

Claim 1 $S \neq \{0\}$

Have G has finite order

$$\Rightarrow x^r = x^s \text{ for } r \neq s \text{ in } \mathbb{Z}$$

$$\Rightarrow \underbrace{r-s}_{\neq 0} \in S$$

$$\Rightarrow S = \underline{\underline{\mathbb{Z}N}} \text{ for some pos int. } N$$

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proved before

$$G = \{1, x, x^2, \dots, x^{n-1}\}$$

has size N

$$\Rightarrow N = n.$$

$$\text{e. } G = \{1, x, x^2, \dots, x^{n-1}\} \text{ and } S = \mathbb{Z}_n. \quad \square$$

Propⁿ: G is cyc of order n , gen'd by x

$$\text{e. } G = \{1, x, x^2, \dots, x^{n-1}\}.$$

Suppose G' a gp, $y \in G'$ with $y^n = 1$.

Then there is a unique homom

$$\varphi: G \rightarrow G' \text{ with } \varphi(x) = y.$$

Pf: $G = \{x^k : k \in \mathbb{Z}\}$

Define φ as $\varphi(x^k) = y^k \quad \forall k \in \mathbb{Z}$.

Claim: φ is well-defined as a map of sets

Pf: Can have $x^r = x^s$ with $r \neq s$.

Must check that if $x^r = x^s$ in G then

$$\varphi(x^r) = \varphi(x^s) \text{ in } G'$$

$$\text{i.e. } y^r = y^s \text{ in } G'$$

$$x^r = x^s \text{ in } G \iff r-s \in S = \mathbb{Z}n$$

$$\implies r = qn + s$$

$$\begin{aligned} \implies y^r &= y^{qn+s} = y^{qn} y^s = (y^n)^q y^s \\ &= 1^q y^s = y^s. \end{aligned}$$

Claim: φ is a homom.

Pf: Need $\varphi(x^a x^b) = \varphi(x^a) \varphi(x^b) \quad \forall a, b \in \mathbb{Z}$.

$$\text{LHS: } \varphi(x^{a+b}) = y^{a+b} \quad \checkmark$$

$$\text{RHS: } \varphi(x^a) \varphi(x^b) = y^a y^b = y^{a+b}$$

Just proved: $\exists \varphi$ homom., $\varphi: G \rightarrow G'$ with $\varphi(x) = y$.

You check: Defining φ with prop $\varphi(x) = y$ determines φ . □

Ex $G =$ any cyclic gp of order n ,
with generator x

$$G' = \text{GL}_2(\mathbb{R})$$

$$y = \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix}$$

Check: $y^n = I$.

Check: $y^n = 1$.

$$\exists \varphi, \text{ homom} \quad \varphi: G \rightarrow GL_2(\mathbb{R})$$
$$x \mapsto y$$

Ex Suppose G is cyc of order 10, gen'd by x
 G' is cyc of order 5, gen'd by y

Then $y^5 = 1 \Rightarrow y^{10} = 1$ so

$\varphi: G \rightarrow G'$ is a well-defined homom.
by $\varphi(x) = y$

Cor Suppose G and G' are cyclic of order n . Then
 $G \cong G'$.
 $\uparrow$ is isomorphic to.

Prf: Suppose $G = \{1, x, x^2, \dots, x^{n-1}\}$
 $G' = \{1, y, y^2, \dots, y^{n-1}\}$

Define $\varphi: G \rightarrow G'$ as $\varphi(x) = y$

Then $\varphi(x^r) = y^r$ for $0 \leq r \leq n-1$

- φ is clearly bijective $\Rightarrow \varphi$ is an isom.
 - φ is a gp homom.
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Particular fact: $\exists$ a surjective gp homom

Particular fact: $\exists$ a surjection gp homom
 $\varphi: S_4 \rightarrow S_3$

- This is special for S_n $n=4,3$

Partition the integers $\{1,2,3,4\}$ into pairs of subsets of size 2

$$\Pi_1: \{1,2\} \cup \{3,4\}$$

$$\Pi_2: \{1,3\} \cup \{2,4\}$$

$$\Pi_3: \{1,4\} \cup \{2,3\}$$

$p \in S_4$ a permutation. Then if j is given
 $p(\Pi_j) = \Pi_k$ for some k .

Ex $p = (1234)$

$$\Pi_1: \{1,2\} \cup \{3,4\}$$



$$\{2,3\} \cup \{4,1\} = \Pi_3$$

$$\Pi_2: \{1,3\} \cup \{2,4\}$$



$$\{2,4\} \cup \{3,1\} = \Pi_2$$

$$\Pi_3: \{1, 4\} \cup \{2, 3\}$$



$$\{2, 1\} \cup \{3, 4\} = \Pi_1$$

$p = (1234)$ induces the permutation

$$\begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{array}$$

We get a map $\varphi: S_4 \rightarrow S_3$ permuting $\{\Pi_1, \Pi_2, \Pi_3\}$

One has : φ satisfies: $\varphi(pq) = \varphi(p)\varphi(q)$
 'ce φ is a sp homom.

' φ satisfies: φ is surjective

$\Rightarrow$ surj. sp hom $\varphi: S_4 \rightarrow S_3$

kernel: $K = \{1, (12)(34), (13)(24), (14)(23)\}$ is $\ker(\varphi)$

$$\underline{(12)(34)}$$

$$\Pi_1: \{1, 2\} \cup \{3, 4\}$$



$$\{2, 1\} \cup \{4, 3\} = \Pi_1$$

$$\Pi_2: \{1, 3\} \cup \{2, 4\}$$



$$\{3, 1\} \cup \{4, 2\} = \Pi_2$$

$$\downarrow$$

$$\{2, 4\} \cup \{1, 3\} = \Pi_2$$

$$\Pi_3: \{1, 4\} \cup \{2, 3\}$$

$$\downarrow$$

$$\{2, 3\} \cup \{1, 4\} = \Pi_3$$

$$\mapsto \varphi((12)(34)) = \tau \in S_3$$